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Significance, interpretation, and quantification of uncertainty in prognostics and remaining useful life prediction

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ABSTRACT

This paper analyzes the significance, interpretation, and quantification of uncertainty in prognostics, with an emphasis on predicting the remaining useful life of engineering systems and components. Prognostics deals with predicting the future behavior of engineering systems, and is affected by various sources of uncertainty. In order to facilitate meaningful prognostics-based decision-making, it is important to analyze how these sources of uncertainty affect prognostics, and thereby, compute the overall uncertainty in the remaining useful life prediction. This paper investigates the classical (frequentist) and subjective (Bayesian) interpretations of uncertainty and their implications on prognostics, and argues that the Bayesian interpretation of uncertainty is more suitable for condition-based prognostics and health monitoring. It is also demonstrated that uncertainty quantification in remaining useful life prediction needs to be approached as an uncertainty propagation problem. Several uncertainty propagation methods are discussed in this context, and the practical challenges involved in such uncertainty quantification are outlined.

1. Introduction

1.1. Prognostics and remaining useful life prediction

Advanced engineering systems are being used for time-critical, safety-critical, and cost-critical missions, and the performance of such engineering systems needs to be monitored through the use of an onboard health management system. An accurate health management system directly aids in diagnosis and prognosis, and eventually facilitates decision-making regarding the operations of such engineering systems. Diagnosis consists of fault detection, isolation, and estimation, while prognosis deals with predicting possible failures and the remaining useful life of these systems.

The prediction of remaining useful life (RUL) is one of the most important functional aspects of prognostics and health management. The RUL prediction is not only necessary to verify whether the mission goal(s) can be accomplished but also important to aid online decision-making activities such as fault mitigation and mission replanning. Sun et al. [1] discuss the benefits of prognostics and explain how the calculation of RUL is important for technical health determination and life extension [2]. Since the prediction of RUL is critical to operations and decision-making, it is imperative that the RUL be estimated accurately. Degradation signals [3,4] and deterioration models [5] have been used in combination with statistical

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methods for estimating remaining useful life. Researchers have investigated both model-based approaches [6] and data-driven approaches [7,8] for prognostics [9] and RUL prediction [10]. A wide variety of advanced computational techniques such as support vector machines [11], artificial neural networks [12] and dynamic Bayesian networks [13] have been used for estimating the remaining useful life of engineering systems. These methods have been applied to a variety of applications including mechanical bearings [14], gears [15], lithium-ion batteries [16], mobile robots [17], helicopter gear plates [18], and unmanned aerial vehicles [19].

1.2. Uncertainty in prognostics

Since prognostics deals with predicting the future behavior of engineering systems, there are several sources of uncertainty that influence such future prediction, and therefore, it is not meaningful to perform prognosis without estimating the associated uncertainty. As a result, uncertainty plays a significant role in estimating the remaining useful life of engineering systems, and therefore, researchers have been developing different types of approaches for quantifying the uncertainty associated with prognostics.

Existing methods for quantifying uncertainty in prognostics and remaining useful life prediction can be broadly classified as being applicable to two different types of situations: testing-based prognostics and condition-based prognostics. Methods for *testing-based* prognostics are based on rigorous testing *before* and/or *after* operating an engineering system (offline), whereas methods for *condition-based* prognostics are based on monitoring the performance of the engineering system *during* operation (online).

There are several research papers that discuss uncertainty quantification in crack growth analysis [20,21], structural damage prognosis [22,23], electronics [24], and mechanical bearings [25], primarily in the context of testing-based approaches. Such approaches may be applicable to smaller components when it may be affordable to run several such components to failure, and it may not be practically feasible to extend this approach to large scale systems. Further, the prediction of remaining useful life is more significant in an online health monitoring context where the performance of a system under operation needs to be monitored and its remaining useful life calculated. Engel et al. [26] discuss several issues involved in the prediction of remaining useful life in online prognostics and health monitoring. Though some of the initial studies on remaining useful life prediction lacked uncertainty measures [27], researchers have recently started investigating the impact of uncertainty on estimating the remaining useful life. For example, there have been several efforts to quantify the uncertainty in remaining useful life of batteries [28] and pneumatic valves [29] in the context of online health monitoring. Different types of sampling techniques [6] and analytical methods [16] have been proposed to predict the uncertainty in the remaining useful life.

However, a review of the aforementioned papers reveals that there exist several challenges in applying uncertainty quantification methods for online health monitoring purposes. For example, several papers claim to account for uncertainty in prognostics using Bayesian filtering techniques like Kalman filtering [30] and particle filtering [31]. Such a claim is not technically accurate because filtering can be used only to estimate the health state of the system based on data. The key in prognostics is to predict future deterioration based on the estimated health state and filtering cannot be used for future prediction. Therefore, it is necessary to resort to other statistical approaches that can compute the uncertainty in the future prediction and the remaining useful life [32].

Another important and related issue is that, while the importance of uncertainty quantification in prognostics has been understood, there have been few efforts to understand and appropriately interpret such uncertainty. Celaya et al. [33] discussed the interpretation of RUL in the context of Kalman filtering-based prognostics techniques, and explained that it is not appropriate to arbitrarily force the variance of RUL to be small. It is necessary to further investigate the aspects of interpretation and quantification of uncertainty, in order to completely understand and quantify the impact and effect of uncertainty on prognostics and remaining useful life prediction.

1.3. Goals and contributions of this paper

The present paper delves deeper into the philosophical aspects of uncertainty in prognostics and focuses on four important questions in order to understand and quantify uncertainty in prognostics.

1. What causes uncertainty in prognostics?
2. How to interpret uncertainty in prognostics?
3. How to facilitate effective treatment of uncertainty in prognostics?
4. How to accurately quantify uncertainty in prognostics?

The answers to the above questions are sought from multiple points of view, and different fundamental concepts relating to uncertainty in prognostics are explained in detail. In order to facilitate better understanding of these fundamental concepts, a conceptual example is presented in Section 2, and used as an illustrating guide, throughout this paper. The answers to the above questions, in turn, lead to the following contributions of this paper.

1. This paper discusses and analyzes the different sources of uncertainty in prognostics in detail. It is explained that the traditional approach for the classification of uncertainty (in terms of "aleatory" versus "epistemic") may not be suitable for prognostics. A new methodology for classification is presented in Section 3.
2. There are several activities that need to be performed in order to account for uncertainty in prognostics and provide useful information for decision-making. While several research papers have confounded uncertainty quantification and uncertainty management, this paper clearly delineates these activities to facilitate better analysis of uncertainty, as explained in Section 4.
3. The interpretation of uncertainty and its implications on prognostics are discussed in detail in Section 5. While statistics and probability methods have been in existence for over 200 years, there has always been a disagreement (amongst mathematicians and statisticians alike) regarding the interpretation of probability. It is important to understand this disagreement before attempting to interpret uncertainty in prognostics; in fact, this disagreement would facilitate better understanding of uncertainty in testing-based prognostics versus condition-based prognostics.
4. A computational framework for uncertainty quantification in prognostics is presented in Section 6, and it is explained that the quantification of uncertainty in RUL needs to be approached as an uncertainty propagation problem. It is explained that analytical expressions for quantifying the uncertainty in RUL are not available even for simple problems involving linear models and Gaussian variables, and that it is necessary to investigate statistical uncertainty propagation methods to accurately quantify the uncertainty in the RUL prediction. Several uncertainty propagation methods are discussed and the challenges involved in quantifying uncertainty in prognostics are outlined.

The scientific arguments presented throughout the paper are substantiated using a practical example in Section 7. Finally, Section 8 concludes the paper and suggests directions for future research.

2. Conceptual example

Consider an engineering component whose health state at any time instant is given by $x(t)$. Consider a simple degradation model, where the rate of degradation of the health state (that decreases with time, due to the presence of damage) is proportional to the current health state. This can be mathematically expressed as

$$\dot{x}(t) \propto x(t), \tag{1}$$

where the constant of proportionality is a negative number. Since differential equations are usually solved by considering discrete time instants, the above equation can be rewritten as

$$x(k+1) = a \cdot x(k) + b, \tag{2}$$

where k represents the discretized time-index. The condition that "the constant of proportionality in Eq. (1) is negative" is equivalent to the condition "$a < 1$ in Eq. (2)". The initial health state, i.e., $x(0)$ is a random variable, and is expressed using a probability distribution. For the sake of illustration, let a denote a measure of the amount of loading on the system (smaller the value of a, higher the degradation rate), and let b denote the parameter of the above degradation model. While a and b are constant and time-invariant (for the sake of illustrating the conceptual example), they are random and expressed using probability distributions. (In practical examples, the probability distributions of a and b could vary as a function of time.)

In order to compute the remaining useful life, it is necessary to choose a threshold function that defines the occurrence of failure. Since $x(k)$ is a decreasing function, the threshold function will indicate that failure occurs when the state value x becomes smaller than a critical lower bound l; the first time instant at which this event occurs indicates the end of life, and this time instant can be used to calculate the RUL. Therefore, the remaining useful life (r, an instance of the random variable R) is equal to the smallest n such that $x(n) < l$. Therefore, the RUL (at the initial time of prediction, i.e., when k=0) can be calculated as

$$r = \inf\{n : x(n) < l\}. \tag{3}$$

Using this conceptual example, this paper explains the significance and interpretation of uncertainty in prognostics. It will be demonstrated that quantifying the uncertainty in the remaining useful life prediction is a non-trivial problem; even when the underlying degradation model is linear (such as the model in Eq. (1)) and the uncertain variables follow Gaussian distributions, closed-form analytical expressions for RUL are not available, and it is impossible to draw conclusions on the distribution type and distribution parameters of RUL.

3. Sources of uncertainty in prognostics

Prognostics is the art of predicting future component/system behavior, identifying possible failure modes, and thereby computing the remaining useful life of the component/system. There are several sources of uncertainty that affect the prediction of future behavior, and in turn, the remaining useful life. In order to aid meaningful prognostics-based decision-making, it is important to analyze how these sources of uncertainty affect prognostics and compute the overall uncertainty in the remaining useful life prediction. In many practical applications, it may even be challenging to identify and individually quantify the different sources of uncertainty that affect prognostics. In some other applications, it may be possible to

quantify a particular source of uncertainty but may not be straightforward to account for its effect in prognostics due to modeling difficulties.

Some researchers have classified the different sources of uncertainty into different categories in order to facilitate uncertainty quantification and management. While it has been customary to classify the different sources of uncertainty into aleatory (arising due to physical variability) and epistemic (arising due to lack of knowledge), such a classification may not be suitable for prognostics in the context of condition-based monitoring and RUL prediction. This is because "true variability" is not present in condition-based monitoring, and this point will be further explored in the rest of this paper. A completely different approach for classification, particularly applicable to condition-based monitoring, is proposed in this paper. These different sources of uncertainty are discussed in detail, in the forthcoming subsections.

3.1. Present uncertainty

Prior to prognosis, it is important to be able to precisely estimate the condition/state of the component/system at the time at which RUL needs to be predicted. Typically, damage (or fault) is expressed in terms of the system/component state (represented using state variables), and therefore, estimating the state is equivalent to estimating the extent of damage (or fault). Such state estimation is commonly addressed using filtering techniques. Output data (usually collected through sensors) is used to estimate the state and many filtering approaches are able to provide an estimate of the uncertainty in the state. In the conceptual example, the state uncertainty is equal to the uncertainty associated with $x(0)$. Practically, it is possible to improve state estimation and thereby reduce this uncertainty by using better sensors and improved filtering approaches.

It is important to understand that the system is at a particular state at any time instant, and the aforementioned uncertainty simply describes the lack of knowledge regarding the "true" state of the system.

3.2. Future uncertainty

The most important source of uncertainty in the context of prognostics is due to the fact that the future is unknown, i.e. the loading, operating, environmental, and usage conditions are not known precisely, and it is important to assess this uncertainty before performing prognosis. In the conceptual example, the future uncertainty is equal to the uncertainty associated with the loading value, i.e., a, from the time of prediction until the time of failure.

If there is no uncertainty regarding the future, then there would be no uncertainty regarding the *true* remaining useful life of the engineering component/system.

3.3. Modeling uncertainty

It is necessary to use a functional degradation model in order to predict future state behavior, i.e. model the response of the system to anticipated loading, environmental, operational, and usage conditions. Further, the end-of-life is also defined using a Boolean threshold functional model, that is used to indicate whether failure has occurred or not. These two models are jointly used to predict the RUL, and they may either be physics-based or data-driven. It may be practically impossible to develop models that accurately predict the underlying reality.

Modeling uncertainty represents the difference between the predicted response and the true response (that can neither be known nor measured accurately), and comprises of several parts: model parameters, model form, and process noise. While it may be possible to quantify these terms until the time of prediction, it is challenging to know their values at future time instants. In the conceptual example in Section 2, Eq. (2) represents the degradation model, $x(n) < l$ represents the Boolean threshold function that indicates failure, b is a model parameter, and the uncertainty in b corresponds to one aspect of modeling uncertainty. Another aspect is the choice of the "linear" form of the model in Eq. (2); the underlying physical phenomenon may differ from this assumption.

3.4. Prediction method uncertainty

Even if all the above sources of uncertainty can be quantified accurately, it is necessary to quantify their combined effect on the RUL prediction, and thereby, quantify the overall uncertainty in the RUL prediction. It may not be possible to do this accurately and this leads to additional uncertainty. For example, when sampling-based approaches are used for prediction, the use of limited number of samples causes uncertainty regarding the estimated probability distribution.

3.5. Summary

The goal in prognostics is to quantify each of the above sources of uncertainty, compute their combined effect on future prediction, and estimate the overall uncertainty in the remaining useful life prediction. It is necessary to identify a systematic approach in order to efficiently process all the above sources of uncertainty, and to this end, the following section discusses four different uncertainty-related activities that need to be performed in order to accurately account for uncertainty in prognostics.

4. Uncertainty-related activities in prognostics

Some research papers in the domain of prognostics and health management discuss uncertainty from representation, quantification, and management points of view [34–38]. While these three are distinctly different processes, they are often confused with one other and used interchangeably. This paper classifies the various tasks related to uncertainty quantification and management into four activities, as explained below. These four tasks need to be performed in order to accurately estimate the uncertainty in the RUL prediction and inform the decision-maker regarding such uncertainty.

4.1. Uncertainty representation and interpretation

The first step is uncertainty representation and interpretation, which in many practical applications is guided by the choice of modeling and simulation framework. For instance, in the conceptual example, $x(0)$, a, and b are uncertain, and it is necessary to choose how to represent this uncertainty. There are several methods for uncertainty representation that differ not only in the level of granularity and detail, but also in how uncertainty is interpreted. Such methods are based on probability theory [39], fuzzy set theory [40,41], evidence theory [42], imprecise probabilities [43], interval analysis [44], etc.

Amongst these theories, probability theory has been widely used in the PHM domain [33]. Using principles of probability, the quantities $x(0)$, a, and b will be represented using probability distributions. Even within the context of probabilistic methods, uncertainty can be interpreted and perceived in two different ways: frequentist (classical) versus subjective (Bayesian). Section 5 outlines the differences between these two schools of thought and argues that the Bayesian approach provides a more suitable interpretation for uncertainty in remaining useful life prediction, in the context of online health monitoring.

4.2. Uncertainty quantification

The second step of uncertainty quantification deals with identifying and characterizing the various sources of uncertainty that may affect prognostics and RUL prediction. While quantities such as $x(0)$, a and b are represented by probability distributions, it is necessary to quantify their statistics and estimate their distribution types and distribution parameters before prognostics can be performed. It is important that these sources of uncertainty are incorporated into models and simulations as accurately as possible. The goal in this step is to address each of these uncertainties separately and quantify them using probabilistic/statistical methods. Note that the Kalman filter, that has been commonly used in prognostics, is essentially a Bayesian tool for uncertainty quantification, where the focus is on quantifying the uncertainty in the states, using health monitoring data.

4.3. Uncertainty propagation

The third step of uncertainty propagation is most relevant to prognostics, since it accounts for all the previously quantified uncertainties and uses this information to predict (1) future states and the associated uncertainty; and (2) remaining useful life and the associated uncertainty. The former is computed by propagating the various sources of uncertainty through the degradation-prediction model. The latter is computed using (1) the uncertainty estimated in the future states along with (2) the previously mentioned Boolean threshold function that is used to indicate end-of-life (EOL). In the conceptual example, the goal would be to account for the uncertainty in $x(0)$, a, and b, and compute the uncertainty in the RUL prediction (Eq. (3)), using the degradation model (Eq. (2)) and the Boolean failure threshold function ($x(n) < l$ in Eq. (3)).

In this step, it is important to understand that the future states and the remaining useful life prediction are simply dependent upon the various uncertainties characterized in the previous step, and therefore, the distribution type and distribution parameters of future states and remaining useful life should not be arbitrarily chosen. Sometimes, a normal (Gaussian) distribution has been assigned to the remaining useful life prediction; such an assignment is erroneous and the true probability distribution of RUL needs to be estimated though rigorous uncertainty propagation of the various sources of uncertainty through the state space model and the EOL threshold function, both of which may be non-linear in practice.

4.4. Uncertainty management

The fourth and final step is uncertainty management, and it is unfortunate that, in several papers, the term “Uncertainty Management” has been used instead of uncertainty quantification and/or propagation. Uncertainty management is a general term used to refer to different activities that aid in managing uncertainty in condition-based maintenance during real-time operation. There are several aspects of uncertainty management. One aspect of uncertainty management attempts to answer the question: “Is it possible to improve the uncertainty estimates?” The answer to this question lies in identifying which sources of uncertainty are significant contributors to the uncertainty in the RUL prediction. For example, if the quality of the sensors can be improved, then it may be possible to obtain a better state estimate (with lesser uncertainty) during Kalman filtering, that may in turn lead to a less uncertain RUL prediction. Another example is when the loading (expressed through a) is a significant contributor of uncertainty; if the uncertainty in RUL needs to be reduced, then it may be necessary

to reduce the uncertainty in a. This is equivalent to mitigating loading conditions in order to reduce the uncertainty in RUL, so that the mission goals may be met. Another aspect of uncertainty management deals with how uncertainty-related information can be used in the decision-making process.

4.5. Summary

Several research papers in the area of prognostics only address the topics of uncertainty quantification and propagation; few papers have directly addressed the topic of uncertainty management. Even within the realm of uncertainty quantification and propagation, the estimates of uncertainty have sometimes been misinterpreted. For example, when statistical principles are used to estimate a parameter, there is an emphasis on calculating the estimate with the minimum variance. When this principle is applied to RUL estimation, it is important not to arbitrarily reduce the variance of RUL itself. For example, when the mean of RUL is being estimated, what needs to be minimized is the *variance of the mean of RUL*, and *not the variance of RUL* itself. Celaya et al. [33] explored this idea and explained that the variance of RUL needs to be carefully calculated by rigorously accounting for the different sources of uncertainty. The present paper delves deeper into this topic of estimating the uncertainty in the remaining useful life prediction. However, prior to uncertainty quantification and propagation, it is necessary to accurately understand and interpret uncertainty.

5. Understanding uncertainty in prognostics

Consider the problem of estimating the uncertainty in prognostics and the remaining useful life prediction. Since probabilistic theory has been predominantly used in the prognostics and health management domain [33], the rest of this paper deals only with probabilistic methods for the treatment of uncertainty.

Researchers have pursued two different classes of methods for prognostics; while the first method is testing-based, the second method is condition-based. While the former is primarily applicable in the context of offline health management, the latter is applicable to online condition-based monitoring and health management. There is a significant difference in the interpretation of uncertainty when these two different approaches are used for prognostics. Understanding this difference is important for prognostics and decision-making, and this is the focus of this section. First, certain basic concepts of probability theory are reviewed and its two most popular interpretations are briefly discussed. Then, the implications of these two interpretations on prognostics are explained in detail.

5.1. Interpreting uncertainty

Though probabilistic methods, mathematical axioms and theorems of probability have been well-established in the literature, there is considerable disagreement among researchers on the interpretation of probability. There are two major interpretations based on physical and subjective probabilities. It is essential to understand the difference between these two interpretations before attempting to interpret the uncertainty in RUL prediction. In fact, the succinct difference between these two interpretations will improve the state-of-the-art in understanding uncertainty in prognostics.

5.1.1. Physical probability

Physical probabilities [45], also referred to objective or frequentist probabilities, are related to random physical experiments such as rolling dice, tossing coins and roulette wheels. Each trial of the experiment leads to an event (that is a subset of the sample space), and in the long run of repeated trials, each event tends to occur at a persistent rate, and this rate is referred to as the "relative frequency". These relative frequencies are expressed and explained in terms of physical probabilities. Thus, physical probabilities are defined only in the context of random experiments. The theory of classical statistics is based on physical probabilities. Within the realm of physical probabilities, there are two types of interpretations: von Mises' frequentist [46] and Popper's propensity [47]; the former is more easily understood and widely used.

The fundamental assumption in this approach is that randomness arises only due to the presence of physical probabilities. If the true value of any particular quantity is deterministic, then it is not possible to associate physical probabilities to that quantity. In other words, when a quantity is not random but unknown, then tools of probability cannot be used to represent this type of uncertainty. For example, the mean of a random variable, sometimes referred to as the population mean, is deterministic. It is meaningless to talk about its probability distribution. In fact, for any type of parameter estimation, the underlying parameter is assumed to be deterministic and only an estimate of this parameter is obtained. The uncertainty in the parameter estimate is addressed through confidence intervals. The interpretation of confidence intervals (as explained later in this section) is sometimes confusing and misleading. Further, the uncertainty in the parameter estimate cannot be used for further uncertainty quantification. For example, if the model parameters of a battery model are estimated under a particular loading condition, then this uncertainty cannot be used for quantifying the battery-response for a similar loading condition.

5.1.2. Subjective probability

Subjective probabilities [48] can be assigned to any "statement". It is not necessary that the concerned statement is in regard to an event that is a possible outcome of a random experiment. In fact, subjective probabilities can be assigned even

in the absence of random experiments. The Bayesian methodology is based on subjective probabilities that are simply considered to be degrees of belief and quantify the extent to which the "statement" is supported by existing knowledge and available evidence. In recent times, the terms "subjective" and "Bayesian" have become synonymous with one another [49]. In the subjective approach, even deterministic quantities can be represented using probability distributions that reflect the subjective degree of the analyst's belief regarding such quantities. As a result, probability distributions can be assigned to parameters that need to be estimated, and therefore, this interpretation facilitates uncertainty propagation after parameter estimation. Interestingly, subjective probabilities can also be applied in situations where physical probabilities are involved [50].

The concept of likelihood and its use in Bayes' theorem are key to the theory of subjective probability. The numerical implementation of Bayes' theorem may be complicated in some practical cases, and several sampling techniques have been developed by researchers to address this issue. Today, Bayesian methods are used to solve a variety of problems in engineering [51]. Filtering techniques such as particle filtering and Kalman filtering are also primarily based on the use of Bayes' theorem and sequential sampling. It must also be noted that such filtering approaches are known as Bayesian tracking methods, not simply because they use Bayes' theorem, but because they provide uncertainty estimates that need to be interpreted subjectively. For example, when Kalman filtering is used to estimate the states of an engineering system, the true states of this system (at any particular time instant) are purely deterministic and there is no "true variability" regarding these states. Therefore, the uncertainty estimate resulting from the use of a Kalman filter is simply indicative of the subjective uncertainty regarding the system states.

5.1.3. Choice of interpretation

Both the frequentist and subjective approaches have been well-established in the literature. In fact, both the approaches may yield similar results (but different interpretations) for a few standard problems involving Gaussian variables. Sometimes, both approaches may be suitable for a given problem at hand; for example, Kalman filtering has a purely frequentist interpretation based on least squares minimization as well as a purely Bayesian interpretation that relies on continuously updating the uncertainty in the state estimates using Bayes' theorem. It is acceptable to interpret uncertainty using the frequentist approach or the Bayesian approach, provided the interpretation is suitable for the problem at hand. The rest of this section explores the suitability of these two interpretations to testing-based prognostics and condition-based prognostics.

5.2. Uncertainty in testing-based prognostics

In testing-based prognostics, the remaining useful life needs to be calculated by testing multiple nominally identical specimens (n samples) of the engineering component/system. Assume that a set of run to failure experiments have been performed with high level of control, ensuring same usage and operating conditions. The time to failure for each of the n samples ($r_i; i=1$ to n) is measured. It is important to understand that *different* RUL values are obtained due to inherent variability across the n different specimens, thereby confirming the presence of physical probabilities. Assume that these random samples belong to an underlying probability density function (PDF) $f_R(r)$, with expected value $E(R)=\mu$ and variance $Var(R)=\sigma^2$. The goal of uncertainty quantification is to characterize this probability density function based on the available n data. Theoretically, an infinite amount of data is necessary to accurately estimate this PDF; however, due to the presence of limited data, the estimated PDF is not accurate. As a result, both frequentist and subjective statisticians compute the uncertainty in the aforementioned estimate, but express and interpret this uncertainty in completely different ways.

For the sake of illustration, assume that the entire PDF can be equivalently represented using its mean and variance; in other words, assume that the random variable R follows a two-parameter distribution. Therefore, estimating the parameters μ and σ is equivalent to estimating the PDF. In the context of physical probabilities (frequentist approach), the "true" underlying parameters μ and σ are referred to as "population mean" and "population standard deviation" respectively. Let $\overline{x}$ and s denote the mean and the standard deviation of the available n data respectively. As stated earlier, due to the presence of limited data, the sample parameters ($\overline{x}$ and s) will not be equal to the corresponding population parameters (μ and σ). The fundamental assumption in this approach is that, since there are true but unknown population parameters, it is meaningless to talk about the probability distribution of any population parameter. Instead, the sample parameters are treated as random variables, i.e., if another set of n data were available, then another realization of $\overline{x}$ and s would have been obtained. Using the sample parameters ($\overline{x}$ and s) and the number of data available (n), frequentists construct confidence intervals on the population parameters (μ and σ).

Confidence intervals can be constructed for both μ and σ [52]. For example, consider the multiple nominally identical specimens of the engineering component that was discussed in Section 2. The term "nominally identical" implies that there is inherent variability in the properties and behavior of these specimens. Suppose that these specimens have been subjected to failure analysis, and their run-to-failure times are available. If the true probability distribution of run-to-failure across multiple specimens is assumed to be Gaussian, the $(1-\alpha)$% confidence interval of the mean run-to-failure time can be calculated as $[\overline{x}-t_{\alpha/2}(s/\sqrt{n}),\ \overline{x}+t_{\alpha/2}(s/\sqrt{n})]$, where $\overline{x}$, s, and n denote the sample mean, sample standard deviation, and number of samples respectively. If the run-to-failure times are given by {100, 105, 98, 110, 92, 97, 85, 120, 93, 101}, then $\mu=100.10$, $s=9.87$, $n=10$, and the 95% confidence interval on the mean run-to-failure is given by [93.98, 106.22]. Using the

properties of the chi-square distribution (χ^2), the confidence interval on the variance can be calculated as

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}, \quad \frac{(n-1)s^2}{\chi^2_{\alpha/2}}\right]$$

For this numerical example, the corresponding confidence interval on the standard deviation is given by [6.79, 18.02].

It is important that these intervals be interpreted correctly. To begin with, the above confidence intervals will decrease as more data is available; therefore, the width of these confidence intervals is simply related to the number of data. The actual uncertainty in the run-to-failure times is given only by the estimate of the standard deviation, and this uncertainty is the result of variability (in material properties, operating conditions, etc.) across all the nominally identical specimens. Further, as stated earlier, the interpretation of confidence intervals may be confusing and misleading. A 95% confidence interval on μ *does not* imply that "the probability that μ lies in the interval is equal to 95%"; such a statement is wrong because μ is purely deterministic and physical probabilities cannot be associated with it. The random variable here is in fact $\overline{x}$, and the confidence interval is calculated using $\overline{x}$. Therefore, the correct interpretation is that "the probability that the estimated confidence interval contains the true population mean is equal to 95%".

Alternatively, it is also possible to address the problem of computing $f_R(r)$ purely from a subjective (Bayesian) point of view. One important difference now is that the Bayesian approach does not clearly differentiate between "sample parameters" and "population parameters". The probability distribution of μ is directly computed using the available data (recall that this was impossible in the frequentist approach since μ is the underlying mean that is precise but unknown), and this uncertainty is referred to as the analyst's degree of belief for the underlying true parameter μ. Similarly, the probability distribution of σ can also be computed using Bayes' theorem. Recall that one realization of the parameters (μ and σ) uniquely defines the PDF $f_R(r)$. However, since the parameters are themselves uncertain, R is now represented by a family of distributions [53,54]. This family of distributions will shrink to the true underlying PDF as the number of available data increases, and the resultant asymptotic PDF (as the number data increases) is simply reflective of the variability (in material properties, operating conditions, etc.) across all the nominally identical specimens.

5.3. Uncertainty in condition-based prognostics

Most of the discussion pertaining to testing-based prognostics is not applicable to condition-based monitoring and prognostics. The distinctive feature of condition-based monitoring is that each component/subsystem/system is considered by itself, and therefore, "variability across specimens" is non-existent. Any such "variability" is spurious and must not be considered. At any generic time instant t_P at which prognostics needs to be performed, the component/subsystem/system is at a specific state. The actual state of the system is purely deterministic, i.e., the true value of each state is completely precise, however unknown. Therefore, if a probability distribution is assigned for this state, then this distribution is simply reflective of the analyst's knowledge regarding this state and cannot be interpreted from a frequentist point of view. Thus, by virtue of definition of condition-based monitoring, physical probabilities are not present here, and a subjective (Bayesian) approach is only suitable for uncertainty quantification.

The goal in condition-based prognostics is, at any generic time instant t_P, to predict the remaining useful life of the component/subsystem/system as a condition-based estimate of the usage time left until failure. First, measurements until time t_P are used to estimate the state at time t_P. Then, using a degradation-prediction model (that may be model-based or data-driven), future state values (corresponding to time instants greater than t_P) are computed. In order to forecast future state values, it is also necessary to assume future loading conditions (and operating conditions), and this is a major challenge in condition-based prognostics. Typically, the analyst subjectively assumes statistics for future loading conditions based on past experience and existing knowledge; thus, the subjective interpretation of uncertainty is clearly consistent across the entire condition-based monitoring procedure, and therefore, inferences made out of condition-based monitoring also need to be interpreted subjectively. The prediction of degradation (forecasting of future state values) is stopped when failure is reached, as indicated by a boolean threshold function that checks whether failure has occurred or not. This indicates the end-of-life (EOL) and the EOL can be directly used to compute the remaining useful life (RUL). Note that it is important to interpret the uncertainty in EOL and RUL subjectively.

For example, consider *one specimen* of the engineering component discussed in Section 2. The uncertainty in $x(0)$, a, and b is related only to the knowledge regarding this particular unit and not an ensemble of units; recall that an ensemble of nominally identical units was considered earlier in Section 5.2. Thus, in condition-based prognostics, it is necessary to meaningfully integrate the degradation equation along with the failure threshold equation, and account for the different sources of uncertainty in $x(0)$, a, and b, and quantify the uncertainty in the remaining useful life. For any given realization of x_0, a, and b, it is possible to compute the first time instant (indicates the end-of-life) at which the failure threshold criterion will be valid, i.e., calculate the smallest value of n at which $x(n) < l$. The challenge is to compute the combined effect of uncertainty in $x(0)$, a, and b on RUL, and estimate the probability distribution of RUL.

5.4. Why is the RUL prediction uncertain?

In light of the above discussion, it is necessary to revisit the question "Why is the RUL uncertain?" from a new perspective. While Section 3 explained that RUL is uncertain because there are several sources of uncertainty that influence

RUL prediction, it is now clear that the uncertainty in RUL could arise due to variability across multiple specimens (testing-based prognostics scenario) or simply due to subjective uncertainty regarding a single specimen (condition-based prognostics scenario). During the online operation of an engineering system, the problem of estimating RUL is relevant only from a condition-based monitoring context, and therefore, all the uncertainty in this regard needs to be interpreted subjectively.

6. Uncertainty quantification in RUL prediction

This section first presents a general computational framework for uncertainty quantification in prognostics and remaining useful life prediction in the context of online, condition-based health monitoring. Second, it illustrates how the problem of computing uncertainty in the remaining useful life prediction can be viewed as an uncertainty propagation problem. Third, the need for rigorous mathematical algorithms for uncertainty quantification in RUL is demonstrated using the conceptual example discussed earlier in Section 2. Fourth, a wide variety of statistical methods for uncertainty propagation are discussed, and the challenges involved in computing the uncertainty in RUL prediction are explained in detail.

6.1. Computational framework for prognostics

Suppose that it is desired to perform prognostics and predict the RUL at a generic time-instant t_P. Daigle and Goebel [55] explain that it is important to develop an architecture for prognostics in engineering systems. This paper considers the architecture in Fig. 1, where the whole problem of prognostics can be subdivided into the following three sub-problems:

1. Present state estimation
2. Future state prediction
3. RUL computation

6.1.1. State estimation

The first step of estimating the state at t_P serves as the precursor to prognosis and RUL computation. Consider the state space model that is used to continuously predict the state of the system as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \boldsymbol{\theta}(t), \mathbf{u}(t), \mathbf{v}(t)) \tag{4}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state vector, $\boldsymbol{\theta}(t) \in \mathbb{R}^{n_\theta}$ is the parameter vector, $\mathbf{u}(t) \in \mathbb{R}^{n_u}$ is the input vector, $\mathbf{v}(t) \in \mathbb{R}^{n_v}$ is the process noise vector, and $\mathbf{f}$ is the state equation. As stated earlier, the state of the system uniquely defines the amount of damage in the system.

The state vector at time t_P, i.e., $\mathbf{x}(t_P)$ (and the parameters $\boldsymbol{\theta}(t_P)$, if they are unknown) is (are) estimated using output data collected until t_P. Let $\mathbf{y}(t) \in \mathbb{R}^{n_y}$, $\mathbf{n}(t) \in \mathbb{R}^{n_n}$, and $\mathbf{h}$ denote the output vector, measurement noise vector, and output equation respectively. Then,

$$\mathbf{y}(t) = \mathbf{h}(t, \mathbf{x}(t), \boldsymbol{\theta}(t), \mathbf{u}(t), \mathbf{n}(t)) \tag{5}$$

Typically, filtering approaches such as Kalman filtering and particle filtering may be used for such state estimation. It must be recalled that these filtering methods are collectively known as Bayesian tracking methods, not only because they use Bayes' theorem for state estimation but also since they rely on the subjective interpretation of uncertainty. In other words, at any time instant, there is nothing uncertain regarding the true states. However, the true states are not known precisely, and therefore, the probability distributions of these state variables are estimated through filtering. The estimated probability distributions are simply reflective of the subjective knowledge regarding those state variables.

Bayes' theorem is used for only state estimation and not thereafter. It may also be noted that there are a few alternatives to Bayesian filtering for the purpose of state estimation; these alternatives are based on least squares-based regression techniques [56], and include methods such as moving least squares [57], total least squares [58] and weighted least squares [59]. However, these methods are based on classical statistics, and express the uncertainty in the states through confidence intervals. Recall from Section 5 that it is not possible to propagate uncertainty that is expressed in the form of confidence intervals, and in prognostics, it is important to facilitate such uncertainty propagation so that the uncertainty in the future

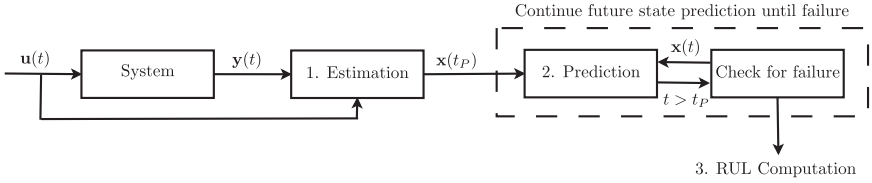

Fig. 1. Model-based prognostics architecture.

states and the remaining useful life can be quantified. That is why Bayesian tracking methods such as Kalman filtering, particle filtering, and their variations have been consistently used for state estimation in different types of engineering applications.

6.1.2. State prediction

Having estimated the state at time t_P, the next step is to predict the future states of the component/system. Note that since the focus is on predicting future, no data is available, and it is necessary to completely rely on and use Eq. (4) for this purpose. This differential equation can be discretized and used to predict the states at any future time instant $t > t_P$, as a function of the states at time t_P.

6.1.3. RUL computation

RUL computation is concerned with the performance of the component that lies outside a given region of acceptable behavior. The desired performance is expressed through a set of n_c constraints, $C_{EOL} = \{c_i\}_{i=1}^{n_c}$, where $c_i: \mathbb{R}^{n_x} \times \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_u} \rightarrow \mathbb{B}$ maps a given point in the joint state-parameter space given the current inputs, $(\mathbf{x}(t), \boldsymbol{\theta}(t), \mathbf{u}(t))$, to the Boolean domain $\mathbb{B} \triangleq [0, 1]$, where $c_i(\mathbf{x}(t), \boldsymbol{\theta}(t), \mathbf{u}(t)) = 1$ if the constraint is satisfied, and 0 otherwise [60].

These individual constraints may be combined into a single *threshold function* $T_{EOL}: \mathbb{R}^{n_x} \times \mathbb{R}^{n_\theta} \times \mathbb{R}^{n_u} \rightarrow \mathbb{B}$, defined as

$$T_{EOL}(\mathbf{x}(t), \boldsymbol{\theta}(t), \mathbf{u}(t)) = \begin{cases} 1, & 0 \in \{c_i(\mathbf{x}(t), \boldsymbol{\theta}(t), \mathbf{u}(t))\}_{i=1}^{n_c} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

T_{EOL} is equal to 1 when any of the constraints are violated. Then, the End of Life (EOL, denoted by E) at any time instant t_P is then defined as the earliest time point at which the value of T_{EOL} becomes equal to one. Therefore,

$$E(t_P) \triangleq \inf\{t \in \mathbb{R}: t \geq t_P \wedge T_{EOL}(\mathbf{x}(t), \boldsymbol{\theta}(t), \mathbf{u}(t)) = 1\}. \tag{7}$$

The Remaining Useful Life (RUL, denoted by R) at time instant t_P is expressed as

$$R(t_P) \triangleq E(t_P) - t_P. \tag{8}$$

Note that the output equation (Eq. (5)) or output data ($\mathbf{y}(t)$) is not used in the prediction stage, and EOL and RUL are dependent only on the state estimates at time t_P; though these state estimates are obtained using the output data, the output data is not used for EOL/RUL calculation after state estimation.

For the purpose of implementation, $\mathbf{f}$ in Eq. (4) is transformed into the corresponding discrete-time version. Discrete time is indexed by k, and there is a one-to-one relation between t and k depending on the discretization level. While the time at which prediction needs to be performed is denoted by t_P, the corresponding index is denoted by k_P. Similarly, let k_E denote the time index that corresponds to the end of life.

6.2. RUL prediction through uncertainty propagation

Thus, it is clear that RUL predicted at time t_P, i.e., $R(t_P)$ depends on

1. Present state estimate ($\mathbf{x}(k_P)$); using the present state estimate and the state space equation in Eq. (4), the future states ($\mathbf{x}(k_P)$, $\mathbf{x}(k_P+1)$, $\mathbf{x}(k_P+2)$, …, $\mathbf{x}(k_E)$) can be calculated.
2. Future loading ($\mathbf{u}(k_P)$, $\mathbf{u}(k_P+1)$, $\mathbf{u}(k_P+2)$, …, $\mathbf{u}(k_E)$); these values are needed to calculate the future state values using the state space equation.
3. Parameter values from time-index k_P until time-index k_E (denoted by $\boldsymbol{\theta}(k_P)$, $\boldsymbol{\theta}(k_P+1)$, …, $\boldsymbol{\theta}(k_E)$).
4. Process noise ($\mathbf{v}(k_P)$, $\mathbf{v}(k_P+1)$, $\mathbf{v}(k_P+2)$, …, $\mathbf{v}(k_E)$).

For the purpose of RUL prediction, all of the above quantities are independent quantities and hence, RUL becomes a dependent quantity. Let $\boldsymbol{X} = \{X_1, X_2, \ldots X_i, \ldots X_n\}$ denote the vector of all of the above independent quantities, where n is the length of the vector $\boldsymbol{X}$, and therefore the number of uncertain quantities that influence the RUL prediction. Then the calculation of RUL (denoted by R) can be expressed in terms of a function as

$$R = G(\boldsymbol{X}) \tag{9}$$

The above functional relation in Eq. (9) can be graphically explained, as shown in Fig. 2. Knowing the values of $\boldsymbol{X}$, it is possible to compute the corresponding value of R, using Fig. 2 that is equivalently represented by Eq. (9). The quantities contained in $\boldsymbol{X}$ are uncertain, and the focus in prognostics is to compute their combined effect on the RUL prediction, and thereby compute the probability distribution of R. The problem of estimating the uncertainty in R is equivalent to propagating the uncertainty in $\boldsymbol{X}$ through G, and it is necessary to use computational methods for this purpose.

6.3. Need for computational approaches

The problem of estimating the uncertainty in R using uncertainty propagation techniques is a non-trivial problem, and needs rigorous computational approaches. This involves estimating the probability density function of R (PDF, denoted by

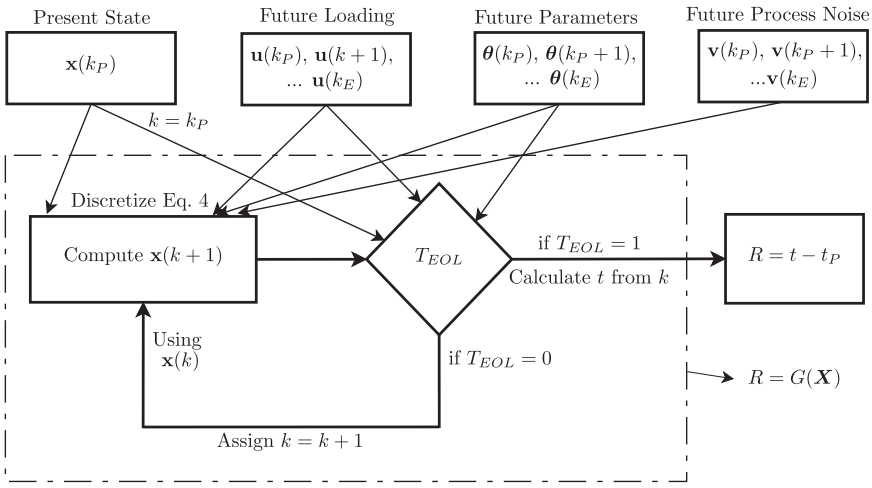

Fig. 2. Definition of *G*.

$f_R(r)$) or equivalently the cumulative distribution function of R (CDF, denoted by $F_R(r)$). In some rare cases, it is possible to analytically obtain the distribution of R. A few of such special cases are listed below:

1. Each and every quantity contained in $\boldsymbol{X}$ follows a normal (Gaussian) distribution, and the function G can be expressed as a weighted linear combination of the quantities in $\boldsymbol{X}$. In this case, R also follows a normal distribution, and its statistics can be calculated analytically.
2. Each and every quantity contained in $\boldsymbol{X}$ follows a lognormal distribution, and if the *logarithm* of the function G can be expressed as a weighted combination of the quantities in $\boldsymbol{X}$, then $\log(R)$ follows a normal distribution whose statistics can be estimated analytically. In other words, R also follows a lognormal distribution.

While Gaussian distributions and linear state space models (linear $\mathbf{f}$ in Eq. (4)) may be used in the prognostics and health management domain, it is important to understand that using linear state-space models is not equivalent to G being linear. In other words, the use of the threshold function along with the linear state-space models automatically renders G non-linear. This can be easily illustrated through the conceptual problem introduced earlier in Section 2.

6.4. Analytical solution of the conceptual problem

In order to illustrate this important point, and to emphasize the importance of using rigorous computational methods, consider the conceptual example from Section 2 whose health degradation is given by $x(k+1) = a \cdot x(k) + b$, where $a < 1$. It can be easily demonstrated that the state value at any future time instant can be expressed as a function of the initial state $x(0)$ as

$$x(n) = a^n \cdot x(0) + \sum_{j=0}^{j=n-1} a^j b \tag{10}$$

Note that $x(n)$ is decreasing and failure happens when $x < l$. Therefore, the remaining useful life (r, an instance of the random variable R) is equal to the smallest n such that $x(n) < l$. Therefore RUL can be calculated as

$$r = \inf\left\{ n : a^n \cdot x(0) + \sum_{j=0}^{j=n-1} a^j b < l \right\}. \tag{11}$$

Assuming that the chosen time-discretization level is infinitesimally small, it is possible to directly estimate the RUL by solving the following equation:

$$a^r \cdot x(0) + \sum_{j=0}^{j=r-1} a^j \cdot b = l. \tag{12}$$

The above equation calculates the RUL (r) as a function of the initial state ($x(0)$) and the generic quantities a and b. Hence, the above equation is similar to G defined earlier in Fig. 2. Even if the only considered source of uncertainty is the state estimate $x(0)$ (that is, a and b are precisely known deterministic constants), RUL (R) follows a Gaussian distribution if and only if it is linearly dependent on $x(0)$. In other words, R follows a Gaussian distribution if and only if Eq. (12) can be

rewritten as

$$\alpha \cdot r + \beta \cdot x(0) + \gamma = 0 \tag{13}$$

for some arbitrary values of α, β, and γ. If it were possible to estimate such values for α, β, and γ, the distribution of RUL can be obtained analytically.

In order to examine if this is possible, rewrite Eq. (12) as

$$x(0) = \frac{1}{a^r}\left(l - \sum_{j=0}^{j=r-1} a^j \cdot b\right). \tag{14}$$

While $x(0)$ is completely on the left hand side of this equation, r appears not only as an exponent in the denominator but is also indicative of the number of terms in the summation on the right hand side of the above equation. Therefore, it is clear that the relationship between r and $x(0)$ is not linear. Therefore, even if the initial state ($x(0)$, a realization of $X(0)$) follows a Gaussian distribution, the RUL (r, a realization of R) does not follow a Gaussian distribution. Furthermore, it is not even possible to analytically estimate the distribution of RUL. Thus, it is clear that, even for a simple problem consisting of a linear state-space model, an extremely simple threshold function, and only one uncertain variable that is Gaussian, the calculation of the probability distribution of R is neither trivial nor straightforward.

6.5. Uncertainty propagation methods for RUL prediction

Practical problems in the prognostics and health management domain may consist of:

1. Several non-Gaussian random variables that affect the RUL prediction,
2. A non-linear multi-dimensional state-space model,
3. Uncertain future loading conditions, and
4. A complicated threshold function that may be defined in multi-dimensional space.

The fact that the distribution of RUL simply depends on the quantities indicated in Fig. 2 implies that it is technically inaccurate to artificially assign the probability distribution type (or any statistic such as the mean or variance) to RUL. It is important to understand that RUL is simply a dependent quantity and the probability distribution of R needs to be accurately estimated using computational approaches.

In order to answer the obvious question: "How to calculate the uncertainty in R and estimate the PDF of R?", it is necessary to resort to rigorous computational methodologies that have been developed by statisticians and researchers for the purpose of uncertainty propagation. These methods can be classified into three types – sampling-based methods, analytical methods, and hybrid methods; while some may calculate the CDF of R, other methods directly generate samples from the probability distribution of R.

6.5.1. Sampling-based methods

The most intuitive method for uncertainty propagation is to make use of Monte Carlo simulation (MCS). The fundamental concept of Monte Carlo simulation [61] is to generate a pseudo-random number that is uniformly distributed on the interval $[0, 1]$; then the CDF of $\boldsymbol{X}$ is inverted to generate the corresponding realization of $\boldsymbol{X}$. Following this procedure, several random realizations of $\boldsymbol{X}$ are generated, and the corresponding random realizations of R are computed. Then the CDF $F_R(r)$ is calculated as the proportion of the number of realizations where the output realization is less than a particular r. The generation of each realization requires one evaluation/simulation of G. Several thousands of realizations may often be needed to calculate the entire CDF, especially for very high/low values of r. Error estimates for the CDF, in terms of the number of simulations, are available in the literature [52]. Alternatively, the entire PDF $f_R(r)$ can be computed based on the available samples of R, using kernel density estimation [62].

For instance, in the conceptual example, if $x(0)$ follows a Gaussian distribution (with mean and standard deviation equal to 975 and 50 respectively), a follows a uniform distribution (with lower and upper bounds of 0.990 and 0.995), and b follows a uniform distribution (with lower and upper bounds of -0.005 and 0 respectively), then the RUL (defined by Eq. (3), where $l=50$) can be calculated as a probability distribution, using Monte Carlo sampling. Using unit discretization (i.e., the time interval between the k^{th} and $(k+1)^{\text{th}}$ instants is equal to one second) for solution, the resultant probability density function (PDF) is shown in Fig. 3.

Several researchers who use particle filtering for state estimation may choose to simply propagate the resultant "particles" into the future; such an approach is only slightly different from Monte Carlo sampling, the difference being that the particles have their own weights. In order to accurately capture the entire probability distribution of RUL, it may be necessary to choose hundreds or thousands of "particles" and such an approach may not be suitable for online health monitoring. That is why it is necessary to investigate other sampling-based methods; these methods are derivations of the basic Monte Carlo algorithm [63,64] but can significantly improve the computational effort. Some of

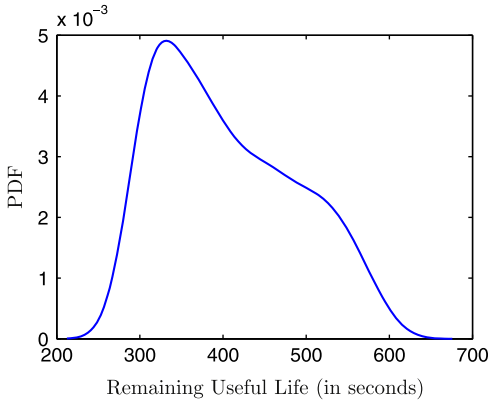

Fig. 3. RUL: conceptual example.

these sampling-based methods are:

1. *Importance sampling*: This algorithm [65] does not generate random realizations of $\boldsymbol{X}$ from the original distribution. Instead, random realizations are generated from a proposal density function, statistics of R are estimated and then corrected based on the original density values and proposal density values.
2. *Adaptive sampling*: This method [66] is an advanced sampling technique where the efficiency of importance sampling is continuously improved by updating the proposal density function based on the information obtained after evaluating G for a few samples of $\boldsymbol{X}$. Two classes of adaptive sampling methods are multi-modal sampling [67] and curvature-based sampling [68]. It has been reported by researchers [52] that adaptive sampling techniques can accurately estimate the tail probabilities using 100–400 samples, while traditional Monte Carlo techniques may require several hundreds of thousands of samples.
3. *Stratified sampling*: In this sampling approach, the overall domain of $\boldsymbol{X}$ is divided into multiple sub-domains and samples are drawn from each sub-domain independently. The process of dividing the overall domain into multiple sub-domains is referred to as stratification. This method is applicable when subpopulations within the overall population are significantly different.
4. *Latin hypercube sampling*: This is a sampling method commonly used in design of computer experiments [69]. When sampling a function of N variables, the range of each variable is divided into M equally probable intervals, thereby forming a rectangular grid. Then, sample positions are chosen such that there is exactly one sample in each row and exactly one sample in each column of this grid. Each resultant sample is then used to compute a corresponding realization of R, and thereby the PDF $f_R(r)$ can be calculated.
5. *Unscented transform sampling*: Unscented transform sampling [6,70] is a sampling approach that focuses on estimating the mean and variance of R accurately, instead of the entire probability distribution of R. Certain pre-determined sigma points are selected in the $\boldsymbol{X}$-space and these sigma points are used to generate corresponding realizations of R. Using weighted averaging principles, the mean and variance of R are calculated.

6.5.2. Analytical methods

A class of analytical methods was developed by researchers in the field of structural engineering, in order to perform uncertainty propagation. These methods facilitate quick (in terms of number of evaluations of G), efficient (reasonably accurate) calculation of the probability distribution of R.

1. *First-order second moment method*: This method [71] uses only the mean and variance of all the uncertain quantities and the first-order Taylor's series expansion of G, in order to calculate the mean and variance of the response quantity R.
2. *First-order reliability method*: This method calculates the CDF function $F_R(r)$ by linearizing G around the so-called most probable point [52,72]. By repeating this calculation for multiple values of r, the entire CDF can be obtained. While this approach is an approximation, it has been proven that it can estimate the CDF with reasonable accuracy in many practical applications [16,52,80].
3. *Inverse first-order reliability method*: This method is the inverse of the first-order reliability method, i.e., it calculates the value of r that corresponds to a given value of β such that $F_R(r) = \beta$. By repeating this approach for several values of β, the entire cumulative distribution function of R can be easily calculated, thereby estimating the uncertainty in R.
4. *Second-order reliability method*: The second-order reliability method [73] improves the estimate of the first-order reliability method through a quadratic approximation of G, instead of a linear approximation. There are different types of quadratic approximations, and corresponding second-order reliability estimates, thereby leading to a variety of computational methods proposed by Kiureghian et al. [73], Tvedt [74], Hohenbichler and Rackwitz [75], etc.

6.5.3. Hybrid methods

In addition to the sampling-based methods and analytical methods, there are also methods that combine both sampling and the use of analytical tools. For example, there are several types of surrogate modeling techniques that have been used by researchers for uncertainty propagation. A few samples of $\boldsymbol{X}$ and the corresponding values of R are computed; these samples are known as training points and training values respectively. Using this information, different types of basis functions are constructed and interpolation is performed in multiple dimensions, in order to facilitate the evaluation of G at untrained locations of $\boldsymbol{X}$. Such surrogate modeling approaches include regression techniques [52], polynomial chaos expansion [76] and kriging [77]. Since each of these methods use different types of basis functions, one may approximate G better than the other.

While the above surrogate modeling methods can be used to directly generate samples of R, Bichon et al. [78] developed a computational methodology to locally estimate the CDF (similar to the first-order reliability method), by emulating G, not globally, but only near the curve represented by the equation $G=r$.

6.5.4. Summary

While the aforementioned uncertainty propagation methods have been used in different types of engineering applications, it is still necessary to investigate their applicability to prognostics. Further, note that uncertainty propagation is still a challenging problem in several practical applications due to the limitations of the aforementioned methods. It is rarely possible to accurately calculate the actual probability distribution of R. Accurate calculation is possible only by using infinite samples for Monte Carlo sampling. Any other method (even the use of a limited, finite number of samples) will lead to uncertainty in the estimated probability distribution, and this additional uncertainty is referred to as prediction-method uncertainty. It is possible to decrease (and maybe, eventually eliminate) this type of uncertainty either by using advanced probability techniques or powerful computing power or both.

6.6. Challenges

There are several challenges in using different uncertainty quantification methods for prognostics, health management and decision-making. It is not only important to understand these challenges but also necessary to understand the requirements of PHM systems in order to efficiently integrate uncertainty quantification along with prognostics and aid risk-informed decision-making. Some of the issues involved in such integration are outlined below:

1. An uncertainty quantification methodology for prognostics needs to be computationally feasible for implementation in online health monitoring. This requires quick calculations, whereas, uncertainty quantification methods are generally time-consuming and computationally intensive.
2. Existing verification, validation, and certification protocols require algorithms to produce deterministic, i.e., repeatable calculations. Several uncertainty quantification methods are non-deterministic, i.e., produce different (albeit, only slightly if implemented well) results on repetition.
3. The uncertainty quantification method needs to be accurate, i.e., the entire probability distribution of $\boldsymbol{X}$ needs to be correctly accounted for, and the functional relationship defined by G in Fig. 2 must not be approximated. Some methods use only a few statistics (usually, mean and variance) of $\boldsymbol{X}$ and some methods make approximations (say for example, linear) of G. Furthermore, it is important to correctly propagate the uncertainty to compute the entire probability distribution of RUL. Sometimes, the probability distribution of RUL may be multi-modal and the uncertainty propagation methodology needs to be able to accurately calculate such distributions.
4. While it is important to be able to calculate the entire probability distribution of RUL, it is also important to be able to quickly obtain bounds on RUL in order to guide online decision-making.

While several uncertainty quantification methods were discussed in this section, a particular method may address one or more of the above issues, and therefore, it may even be necessary to resort to different methods to achieve different goals. Future research needs to continue this investigation, analyze different types of uncertainty quantification methods and study their applicability to prognostics before such methods can be applied in practice.

7. Numerical illustration: Power system of a UAV

In order to illustrate the importance of uncertainty quantification in prognostics and online health monitoring, this section considers the power system of an unmanned aerial vehicle [79], that is being used as a test-bed for prognostics and decision-making at NASA Langley and NASA Ames Research Centers.

7.1. Description of the model

A lithium-ion battery [80] is used to power the unmanned aerial vehicle and this battery is modeled using an electrical equivalent circuit, as shown in Fig. 4. In this circuit, the large capacitance C_b holds the charge q_b of the battery. The nonlinear C_b captures the open-circuit potential and concentration overpotential. The R_{sp}–C_{sp} pair captures the major nonlinear

voltage drop due to surface overpotential, R_s captures the so-called Ohmic drop, and R_p models the parasitic resistance that accounts for self-discharge. This empirical battery model is sufficient to capture the major dynamics of the battery, but ignores temperature effects and other minor battery processes.

State-space models can be constructed using the equivalent circuit model, and they are used for RUL prediction with a discrete time-step of 1 s.

The state-of-charge, *SOC*, is computed as

$$SOC = 1 - \frac{q_{max} - q_b}{C_{max}}, \tag{15}$$

where q_b is the current charge in the battery (related to C_b), q_{max} is the maximum possible charge, and C_{max} is the maximum possible capacity. The resistance related to surface overpotential is a nonlinear function of *SOC*:

$$R_{sp} = R_{sp_0} + R_{sp_1} \exp(R_{sp_2}(1 - SOC)), \tag{16}$$

where R_{sp_0}, R_{sp_1}, and R_{sp_2} are empirical parameters. The resistance, and, hence, the voltage drop, increases exponentially as *SOC* decreases.

The capacitance C_b is expressed as a third-order polynomial function of *SOC*:

$$C_b = C_{b_0} + C_{b_1} SOC + C_{b_2} SOC^2 + C_{b_3} SOC^3 \tag{17}$$

It is of interest to predict the end-of-discharge, that is defined to occur when the battery voltage is less than a threshold V_{EOD}. The remaining useful life (RUL) of the battery is indicative of the time until end-of-discharge (EOD). The numerical details of the electrical circuit model are provided in Table 1.

The following sub-sections discuss the various sources of uncertainty and estimate the uncertainty in the remaining useful life (RUL).

7.2. Sources of uncertainty

The different sources of uncertainty considered in this case study are listed below.

1. *Loading uncertainty*: Saha et al. [79] quantified the uncertainty in the loading for a battery used to power an unmanned aerial vehicle. Several flight segments were identified and the loading was quantified in each segment. In order to illustrate the importance of accurate uncertainty quantification, two different loading cases are considered in this paper.

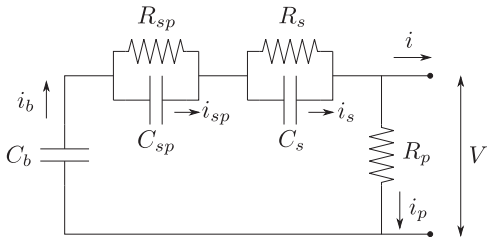

Fig. 4. Battery equivalent circuit.

Table 1
Battery model parameters.

Parameter	Value	Unit
C_{b_0}	19.80	Farad (F)
C_{b_1}	1745.00	Farad (F)
C_{b_2}	−1.50	Farad (F)
C_{b_3}	−200.20	Farad (F)
R_s	0.0067	Ohm (Ω)
C_s	115.28	Farad (F)
R_p	1×10^4	Ohm (Ω)
C_{sp}	316.69	Farad (F)
R_{sp_0}	0.0272	Ohm (Ω)
R_{sp_1}	1.087×10^{-16}	Ohm (Ω)
R_{sp_2}	34.64	No unit
q_{max}	3.11×10^4	Coulomb (C)
C_{max}	30 807	Coulomb (C)
V_{EOD}	16	Volt (V)

First, a constant amplitude loading is considered and the constant amplitude is chosen to be random; the constant amplitude (in Amps) is considered to be normally distributed ($N(35, 5)$), and this distribution is truncated at a specified lower bound (5.0) and upper bound (80). Second, a variable amplitude loading scenario is considered with six segments; within each segment, the amplitude is considered constant. The duration (T, in seconds) and the amplitude (current, I, in Amps) of each segment are considered to be random; therefore, there are 12 random variables, each of which is assumed to follow a truncated normal distribution. The statistics are provided in Table 2.

2. *State uncertainty*: State estimation is addressed using a filtering technique that can continuously estimate the uncertainty in the state based on the available measurements. There are three state variables ((1) charge in C_b; (2) charge in C_{sp}; and (3) charge in C_s), and their mean values are shown in Figs. 5–7. The co-efficient of variation (CoV, defined as the ratio of standard deviation to mean) is chosen to be equal to 0.1 for the purpose of illustration.
3. *Process noise uncertainty*: At any time instant, there are three states, and hence three process noise terms. All the three process noise terms are assumed to have zero mean and variances equal to 1, 1×10^{-4}, and 1×10^{-6}.

As described in detail in Section 6, the goal in prognostics is to accurately account for these different sources of uncertainty and compute the uncertainty in the RUL prediction. Monte Carlo simulation is used for uncertainty quantification and the results are used to discuss some of the challenges involved in applying uncertainty quantification methods to online health monitoring and prognostics.

7.3. Constant amplitude loading: results

The uncertainty in the end-of-discharge prediction is computed continuously as a function of time, and the corresponding PDFs at multiple time instants can be seen in Figs. 8 and 9.

As it can be seen clearly, the shape of the probability density function significantly changes near failure, from a bell shaped distribution to a triangular distribution. It is particularly important to be able to accurately predict the RUL near failure. If any distribution type (say, Gaussian) had been assumed for initial predictions, then such assumptions would clearly be violated. Instead of assigning distribution types to EOD and RUL, the prediction methodology should simply treat them as dependent quantities and use uncertainty propagation methods to estimate their uncertainty.

7.4. Variable amplitude loading: results

While the constant amplitude loading scenario illustrated the curse of assuming distribution types for the EOD (and hence, RUL), the variable amplitude loading scenario provides certain new insights. Consider the initial prediction for the end-of-discharge (EOD) of the battery and calculate its probability distribution using Monte Carlo sampling. Multiple

Table 2
Variable amplitude loading: statistics.

Segment	I_μ	I_σ	I_l	I_u	T_μ	T_σ	T_l	T_u
Take-off	80	7	70	100	60	10	50	75
Climb	30	5	22	40	120	10	90	140
Cruise	15	3	10	22	90	10	70	115
Turn	35	5	25	47	120	10	100	145
Glide	5	1	2	8	90	10	75	120
Landing	40	5	30	53	60	10	40	80

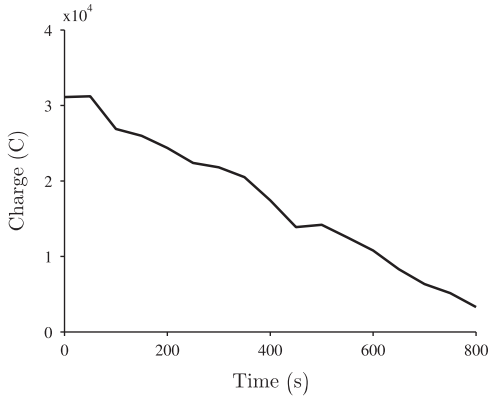

Fig. 5. State no. 1: charge in C_b.

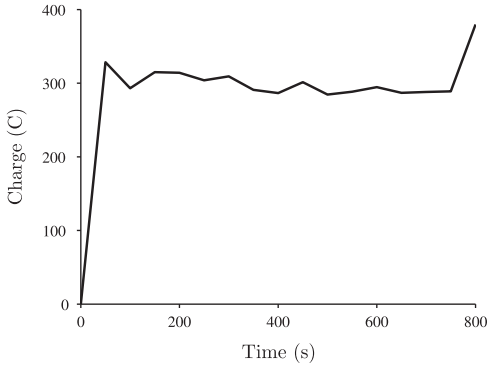

Fig. 6. State no. 2: charge in C_{sp}.

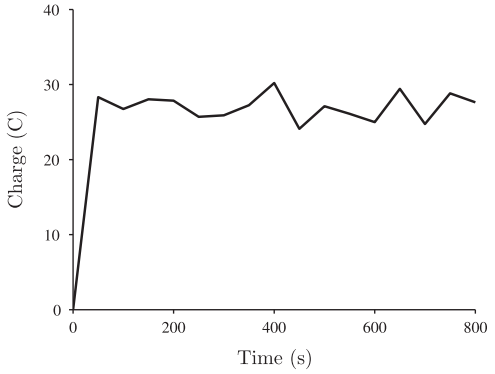

Fig. 7. State no. 3: charge in C_s.

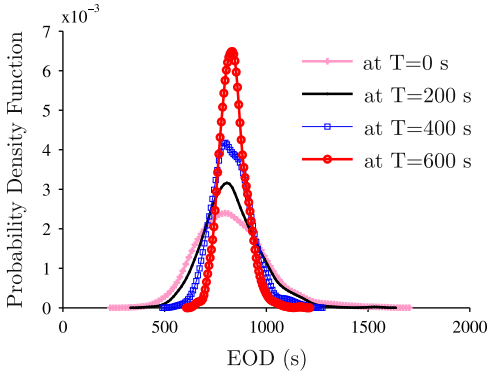

Fig. 8. EOD prediction at multiple time instants.

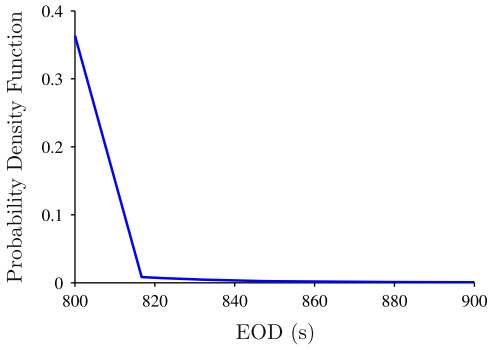

Fig. 9. EOD prediction at $T=800$ s (near failure).

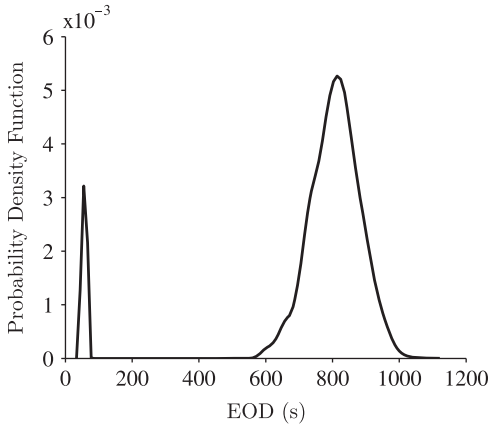

Fig. 10. EOD prediction: variable amplitude loading scenario.

realizations of each source of uncertainty are used to compute the corresponding realizations of EOD that can be used to estimate the probability density function (PDF) of EOD, as shown in Fig. 10.

Note that the above distribution is multi-modal in nature. The two modes do not correspond to multiple failure modes (note that there is only one failure mode for this example, that is specified in terms of battery voltage) but are simply related to the chosen statistics for the future loading conditions. Therefore, it is challenging to identify such multi-modal nature without rigorous Monte Carlo sampling.

7.5. Discussion

The case study presented in this section illustrated the following points:

1. It is important not to arbitrarily assign statistical properties (such as distribution type, mean and standard deviation) for the remaining useful life prediction.
2. The shape of the distribution of RUL may significantly change during the course of the operation of the engineering system.
3. The distribution of RUL may have multiple modes and it is important to accurately capture such modes so that this distribution may be useful for decision-making.

In summary, it is important to capture all the characteristics of the probability distribution of RUL (that is based on the end-of-discharge in this case study), and this can be accomplished only by using accurate uncertainty quantification methodologies without making critical assumptions regarding the PDF of the RUL (shape, mean, median, mode, standard deviation, etc.). The goal must be to accurately calculate the probability distribution of R by propagating the different sources of uncertainty through G as indicated in Fig. 2. While computationally extensive Monte Carlo sampling can achieve this goal with reasonable accuracy, it may not be suitable for online prognostics and health monitoring since Monte Carlo sampling is time-consuming. It is necessary to investigate whether other computational approaches (that were discussed earlier in Section 6.5) are suitable for this purpose. This investigation will be continued in the future and used to guide further research.

8. Conclusion

This paper discussed the significance and interpretation of uncertainty in the context of prognostics and health management. The prediction of remaining useful life in engineering systems is affected by several sources of uncertainty, and it is important to correctly interpret this uncertainty in order to facilitate meaningful decision-making. Uncertainty can be interpreted in two ways, either in terms of physical probabilities from a frequentist point of view or in terms of subjective probabilities from a Bayesian point of view. While a frequentist interpretation may be suitable for testing-based prognostics, physical probabilities are not present in the context of condition-based prognostics. Therefore, uncertainty in online, condition-based health monitoring needs to be interpreted subjectively, and hence, a Bayesian approach is more suitable for this purpose. It was also explained that Bayesian tracking methods for state estimation are so-called not only because they use Bayes' theorem but also because they are based on the principle of subjective probability.

This paper also emphasized the importance of accurately computing the uncertainty in the remaining useful life prediction. It was illustrated that it is not possible to analytically calculate the uncertainty in the remaining useful life

prediction even for certain simple problems involving Gaussian random variables and linear state-space models (used for degradation-prediction). Therefore, it is necessary to resort to computational methodologies for such uncertainty quantification and compute the probability distribution of remaining useful life prediction. In this process, it is important not to make assumptions regarding the shape of the probability distribution of the remaining useful life prediction or any of its statistics such as the mean, median and standard deviation.

Finally, it was explained that the problem of estimating the probability distribution of remaining useful life can be viewed as an uncertainty propagation problem that can be solved using different types of computational approaches. Several sampling-based methods, analytical methods and hybrid methods have been developed by researchers in the field of uncertainty quantification and it is necessary to investigate the applicability of these methods to prognostics and health management. Further, several challenges involved in integrating uncertainty quantification techniques into prognostics and health management were outlined. It is clear that further research is necessary to address these challenges and develop a comprehensive framework for uncertainty quantification in prognostics and health management.

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